

On Transiso Graphs of Groups of order less than 32

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Abstract

Transiso graph $\Gamma_d(G)$ is defined in [6] for a finite group G and a divisor d of $|G|$. In the present paper, we have determined some finite groups G for which the graphs $\Gamma_d(G)$ are complete for each divisor d of $|G|$. We have also discussed the completeness of transiso graphs for the groups of order less than 32.

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1 Introduction

Let G be a finite group and H be a subgroup of G . A *normalized right transversal (NRT)* S of H in G is a subset of G obtained by selecting one and only one element from each right coset of H in G and $1 \in S$. An NRT S has an induced binary operation \circ given by $\{x \circ y\} = S \cap Hxy$, with respect to which S is a right loop with identity 1 (see [9, Proposition 2.2, p.42], [7]). Conversely, every right loop can be embedded as an NRT in a group with some universal property (see [7, Theorem 3.4, p.76]). Let $\langle S \rangle$ be the subgroup of G generated by S and H_S be the subgroup $\langle S \rangle \cap H$. Then, $H_S = \langle \{xy(x \circ y)^{-1} | x, y \in S\} \rangle$ and $H_S S = \langle S \rangle$. Identifying S with the set $H \backslash G$ of all right cosets of H in G , we get a transitive permutation representation $\chi_S : G \rightarrow \text{Sym}(S)$ defined by $\{\chi_S(g)(x)\} = S \cap Hxg$, $g \in G, x \in S$. The kernel $\ker \chi_S$ of this action is $\text{Core}_G(H)$, the core of H in G . The group $G_S = \chi_S(H_S)$ is known as the *group torsion* of the right loop S (see [7, Definition 3.1, p.75]) which depends only on the right loop structure \circ on S and not on the subgroup H . Since χ_S is injective on S and if we identify S with $\chi_S(S)$, then $\chi_S(\langle S \rangle) = G_S S$ which also depends only on the right loop S and S is an NRT of G_S in $G_S S$. One can also verify that $\ker(\chi_S|_{H_S S} : H_S S \rightarrow G_S S) = \ker(\chi_S|_{H_S} : H_S \rightarrow G_S) = \text{Core}_{H_S S}(H_S)$ and $\chi_S|_S = \text{the identity map on } S$. If H is a corefree subgroup of G , then there exists an NRT T of H in G which generates G (see [2]). In this case, $G = H_T T \cong G_T T$ and $H = H_T \cong G_T$. Also (S, \circ) is a group if and only if G_S is trivial. Let $\mathcal{T}(G, H)$ denote the set of all normalized right transversals (NRTs) of H in G . Two NRTs $S, T \in \mathcal{T}(G, H)$ are said to be *isomorphic* (denoted by $S \cong T$), if their induced right loop structures are isomorphic. A subgroup H is normal in G if and only if all NRTs of H in G are isomorphic to G/H (see [7]).

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Throughout the paper, we shall assume that G is a finite group and d is a divisor of the order $|G|$ of the group G . Let $V_d(G)$ be the set of all subgroups of G of order d . We define a graph $\Gamma_d(G) = (V_d(G), E_d(G))$ with $\{H_1, H_2\} \in E_d(G)$ if and only if there exists $S_i \in \mathcal{T}(G, H_i)$ ($i = 1, 2$) such that $S_1 \cong S_2$ with respect to the right loop structure induced on S_i . We will call this graph a *transiso graph* (see [6]). If G has no subgroup of order d , then $\Gamma_d(G)$ is a null graph (a graph having empty vertex set and empty edge set). If G has unique subgroup of order d , then $\Gamma_d(G)$ is an empty graph (a graph having empty edge set). We shall denote transiso graph $\Gamma_d(G)$ by Γ_d if there is no confusion about G . A group G is called a *t-group* if $\Gamma_d(G)$ is a complete graph for each divisor d of $|G|$.

In this paper, we have determined all t-groups of the order less than 32. In the Section 2, we have recalled some preliminary results related to transiso graph from the paper [6]. We have also discussed about the relation of adjacency and proved that the direct product of two t-groups of coprime order is a t-group. In the Section 3, we have discussed about the transiso graphs of some non-abelian groups like dicyclic groups, quasidihedral groups and the groups of the order pq , $4p$, $2pq$ and $2p^2$ for distinct odd primes p and q . We have classified all the t-groups of order less than 32 in the Section 4 and prepared tables and figures for these groups in the Section 5.

2 Preliminaries

We first recall some basic results from the paper [6] and prove some elementary results which will be used in the present paper.

Proposition 2.1. [6, Proposition 2.1] *A subgroup of a group G is always adjacent with its automorphic images in $\Gamma_d(G)$ for any divisor d of $|G|$.*

Proposition 2.2. [6, Proposition 2.2] *Let H_1 and H_2 be corefree subgroups of G . Let $S_i \in \mathcal{T}(G, H_i)$ ($i = 1, 2$) such that $S_1 \cong S_2$ and $\langle S_i \rangle = G$. Then, an isomorphism between S_1 and S_2 can be extended to an automorphism of G which sends H_1 onto H_2 .*

Proposition 2.3. [6, Proposition 2.3] *A finite abelian group G is a t-group if and only if each sylow subgroup of G is either elementary abelian or cyclic.*

Corollary 2.4. [6, Corollary 2.2] *An elementary abelian group is a t-group.*

A very simple observation shows that a finite cyclic group also is a t-group.

Proposition 2.5. [6, Proposition 2.4] *The dihedral group D_{2n} of order $2n$ is a t-group.*

One can easily observe that the number of vertices in the graph $\Gamma_d(D_{2n})$ is equal to the number of subgroups of D_{2n} of order d and is given by

$$V_d(D_{2n}) = \begin{cases} 1 & \text{if } \frac{2n}{d} \text{ is odd.} \\ \frac{2n}{d} & \text{if } \frac{2n}{d} \text{ is even and does not divide } n. \\ \frac{2n}{d} + 1 & \text{if } \frac{2n}{d} \text{ is even and divides } n. \end{cases}$$

Proposition 2.6. [6, Proposition 3.1] *Let G be a non p -central finite p -group. Then, $\Gamma_p(G)$ is complete if and only if whenever H is a non-normal subgroup of G of order p , $G \cong H \rtimes K$ for some subgroup K of G with $G/L \cong K$ for any normal subgroup L of G of order p .*

Proposition 2.7. [6] *Let p be an odd prime and G be a non-abelian group. Then,*

1. If the group G is a t -group and $|G| = p^3$, then G is of exponent p .
2. If $|G| = p^4$, then $\Gamma_p(G)$ is not a complete graph.
3. If $|G| = p^5$ and $\Gamma_p(G)$ is complete, then $\Phi(G) = G' = Z(G) \cong C_p \times C_p$.

Let G be a finite group and d be a divisor of $|G|$. Let us define a relation \sim_d on the set $V_d(G)$ of all subgroups of the group G of order d such that two subgroups H_1 and H_2 are related by the relation \sim_d if either $H_1 = H_2$ or H_1 and H_2 are adjacent in the graph $\Gamma_d(G)$. We call this relation \sim_d the relation of adjacency in the graph $\Gamma_d(G)$. It is trivial that the relation \sim_d is reflexive and symmetric on $V_d(G)$.

Proposition 2.8. *If the relation \sim_d defined above is a transitive relation on $V_d(G)$, then $\Gamma_d(G)$ is either a complete graph or a disjoint union of complete graphs.*

Proof. Assume that the relation \sim_d is a transitive relation on $V_d(G)$. Then, it is an equivalence relation on $V_d(G)$ and hence it gives a partition of $V_d(G)$ and each component of this partition corresponds to a complete graph. \square

Lemma 2.9. *Let H_i and K_i (for $i = 1, 2$) be subgroups of the groups G_i such that there exist NRTs $S_i \in \mathcal{T}(G_i, H_i)$ and $T_i \in \mathcal{T}(G_i, K_i)$ with $S_i \cong T_i$. Then, $S_1 \times S_2 \cong T_1 \times T_2$.*

Proof. One can easily observe that $S_1 \times S_2 \in \mathcal{T}(G_1 \times G_2, H_1 \times H_2)$, for an element $(g_1, g_2) \in G_1 \times G_2$ can be expressed as $(g_1, g_2) = (h_1 s_1, h_2 s_2) = (h_1, h_2)(s_1, s_2)$ where $h_i \in H_i$ and $s_i \in S_i$. Similarly $T_1 \times T_2 \in \mathcal{T}(G_1 \times G_2, K_1 \times K_2)$. Then, the map $f \times g : S_1 \times S_2 \rightarrow T_1 \times T_2$ given by $(s, t) \mapsto (f(s), g(t))$, is a right loop isomorphism where $f : S_1 \rightarrow T_1$ and $g : S_2 \rightarrow T_2$ are right loop isomorphisms. \square

Proposition 2.10. *The direct product of two t -groups of coprime order is a t -group.*

Proof. Let G_1 and G_2 be two t -groups of coprime order. Let $G = G_1 \times G_2$ and H, K be subgroups of G of same order. Then, $H = H_1 \times H_2$ and $K = K_1 \times K_2$ for some subgroups $H_1, K_1 \leq G_1$, $H_2, K_2 \leq G_2$ such that $|H_1| = |K_1| = d_1$ and $|H_2| = |K_2| = d_2$, for a subgroup of the direct product of groups of coprime order is the direct product of subgroups of corresponding groups (see [10, Corollary, p.141]). Since G_1 and G_2 are t -groups, $H_1 \sim_{d_1} K_1$ and $H_2 \sim_{d_2} K_2$. Therefore by Lemma 2.9, the subgroups H and K are adjacent in the corresponding transiso graph. Hence the group G is also a t -group. \square

3 Transiso Graphs for some Non-abelian Groups

In this section, we have determined transiso graphs for some non-abelian groups like dicyclic groups, quasidihedral groups and the groups of the order pq , $4p$, $2pq$ and $2p^2$ for distinct odd primes p and q .

The *dicyclic group* (or binary dihedral group) $Q_{4n} = \langle a, b \mid a^{2n}, a^n b^2, abab^{-1} \rangle$ is a group of order $4n$ for $n \geq 1$ (see [8, p.347]). It is a non-abelian group for $n > 1$ and it is a cyclic group for $n = 1$ (i.e. $Q_4 \cong C_4$). A generalized quaternion group is a special case of the dicyclic group Q_{4n} when $n = 2^k$ for some positive integer k .

In order to prove the Proposition 3.2, we need the following elementary lemma.

Lemma 3.1. *A subgroup of the dicyclic group Q_{4n} is either cyclic or dicyclic. Moreover if d is a divisor of $4n$, then*

1. *There is unique subgroup (namely $\langle a^{\frac{2n}{d}} \rangle$) of Q_{4n} of order d if 4 does not divide d .*
2. *There are i subgroups $(\langle a^i, a^j b \rangle, 0 \leq j < i)$ of order d conjugate to each other if 4 divides d and $i = \frac{4n}{d}$ is odd.*
3. *A subgroup of order d is either $\langle a^{\frac{i}{2}} \rangle$ or conjugate to one of $\langle a^i, b \rangle$ or $\langle a^i, ab \rangle$ if 4 divides d and $i = \frac{4n}{d}$ is even.*

Proof. Let H be a nontrivial proper subgroup of Q_{4n} of the order d . Clearly $\langle a \rangle$ is maximal cyclic subgroup of Q_{4n} of index 2. The composite homomorphism $H \hookrightarrow Q_{4n} \rightarrow Q_{4n}/\langle a \rangle$ is either trivial or onto with the kernel $H \cap \langle a \rangle = \langle a^i \rangle$ for unique divisor i of $2n$. If the homomorphism is trivial, then $H = H \cap \langle a \rangle = \langle a^i \rangle$ for unique divisor $i = \frac{2n}{d}$ of $2n$. Therefore the subgroup H is cyclic in this case. Now if the homomorphism is onto, then $H/\langle a^i \rangle \cong Q_{4n}/\langle a \rangle \cong C_2$. Since $H \not\subseteq \langle a \rangle$, H has an element $a^j b$ and $a^n \in \langle a^i \rangle$ for $(a^j b)^2 = a^n \in H$. Therefore $H \cap \langle a \rangle = \langle a^i \rangle$ for unique divisor $i = \frac{4n}{d}$ of n . Now we have an appropriate element $a^j b \in H - \langle a \rangle$ where $0 \leq j < i$, such that $H = \langle a^i, a^j b \rangle$. Clearly H is a dicyclic group (precisely $H \cong Q_{4, \frac{n}{i}}$) for $(a^i)^{\frac{d}{2}} = 1$, $(a^i)^{\frac{d}{4}} = (a^j b)^2$ and $(a^j b)a^i(a^j b)^{-1} = (a^i)^{-1}$.

Now we prove the next part of the lemma.

Let H be a subgroup of Q_{4n} of order d and $i = \frac{4n}{d}$.

If d is not a multiple of 4, then there is no subgroup of Q_{4n} of order d which is dicyclic and so $H = \langle a^{\frac{i}{2}} \rangle$ is a cyclic subgroup.

If d is a multiple of 4, then there are two cases.

If $d \nmid 2n$ i.e. i is odd, then H can not be contained in $\langle a \rangle$ so H is dicyclic subgroup of the form $\langle a^i, a^j b \rangle$. If $i \leq j$, then we can find l such that $0 \leq l < i$ and $H = \langle a^i, a^l b \rangle$. So we can conclude that $0 \leq j < i$ and hence there are i subgroups of order d which are conjugates.

If $d \mid 2n$ i.e. i is even, then H is either $\langle a^{\frac{i}{2}} \rangle$ or of the form $\langle a^i, a^j b \rangle$. Using above arguments, we can see that there are $\frac{i}{2}$ subgroups conjugate to $\langle a^i, b \rangle$ and $\frac{i}{2}$ subgroups conjugate to $\langle a^i, ab \rangle$. \square

One can easily observe that an abelian normal subgroup of the group Q_{4n} is cyclic subgroup contained in the maximal cyclic subgroup and a non-abelian normal subgroup of Q_{4n} has index less than or equal to 2.

Proposition 3.2. *The dicyclic group $Q_{4n} = \langle a, b \mid a^{2n}, a^n b^{-2}, abab^{-1} \rangle$ of order $4n$ is a t -group.*

Proof. Let d be a divisor of $4n$ and $i = \frac{4n}{d}$.

First assume that $4 \nmid d$. Then by Lemma 3.1, there is unique subgroups of Q_{4n} of order d and so $\Gamma_d(Q_{4n})$ is trivially a complete graph.

Now assume that $4 \mid d$ and i is odd. Then by Lemma 3.1, there are i subgroups of order d conjugate to $\langle a^i, b \rangle$ and so $\Gamma_d(Q_{4n})$ is a complete graph.

Finally assume that $4 \mid d$ and i is even. Then, a subgroup of order d is either $H_1 = \langle a^{\frac{i}{2}} \rangle$ or conjugate to exactly one of $H_2 = \langle a^i, b \rangle$ or $H_3 = \langle a^i, ab \rangle$. Note that H_1 is a normal subgroup of Q_{4n} and so its all NRTs are isomorphic to $Q_{4n}/H_1 (\cong D_{2, \frac{i}{2}})$.

Now choose $S_2 = \{a^{2j+k}b^k \mid 0 \leq j < \frac{i}{2}, k = 0, 1\}$ in $\mathcal{T}(Q_{4n}, H_2)$ and $S_3 = \{a^{2j}b^k \mid 0 \leq j < \frac{i}{2}, k = 0, 1\}$ in $\mathcal{T}(Q_{4n}, H_3)$. Note that $\langle S_2 \rangle = \langle a^2, ab \rangle$ and $\langle S_3 \rangle = \langle a^2, b \rangle$. Then, $H_{S_2} = \langle S_2 \rangle \cap H_2 = \langle a^i \rangle \leq \langle S_2 \rangle$ and $H_{S_3} = \langle S_3 \rangle \cap H_3 = \langle a^i \rangle \leq \langle S_3 \rangle$. Therefore $G_{S_2} = G_{S_3} = \{1\}$ and hence S_2 and S_3 are groups.

Let \circ_2 denote the induced binary operation on S_2 as described in the Section 1. One can observe that, $(a^2)^{\frac{i}{2}} = (ab)^2 = (ab \circ_2 a^2)^2 = 1$. This implies that $S_2 \cong D_{2, \frac{i}{2}}$. One can similarly observe that $S_3 \cong D_{2, \frac{i}{2}}$. This shows that the graph $\Gamma_d(Q_{4n})$ is complete. \square

It follows from the Lemma 3.1 that the number of vertices in the graph $\Gamma_d(Q_{4n})$ can be given by

$$V_d(Q_{4n}) = \begin{cases} 1 & \text{if 4 does not divide } d. \\ \frac{4n}{d} & \text{if 4 divides } d \text{ and } \frac{4n}{d} \text{ is odd.} \\ \frac{4n}{d} + 1 & \text{if 4 divides } d \text{ and } \frac{4n}{d} \text{ is even.} \end{cases}$$

The quasidihedral (or semidihedral) group $QD_{2^n} = \langle a, b \mid a^{2^{n-1}}, b^2, baba^{2^{n-2}+1} \rangle$ is a non-abelian group of order 2^n where $n \geq 4$ (see [5, p.191]). Its subgroup structure can be given by the following lemma.

Lemma 3.3. *A proper nontrivial subgroup of the quasidihedral group QD_{2^n} is either cyclic or dihedral or generalized quaternion.*

Proof. The proof is similar to that of the Lemma 3.1. \square

From [5, Theorem 4.10, p.199], it follows that an abelian normal subgroup of the quasidihedral group QD_{2^n} of order $d = 2^m$ is cyclic (precisely $\langle a^{2^{n-m-1}} \rangle$) and a non-abelian normal subgroup of QD_{2^n} has index less than or equal to 2.

Now we have the following proposition from which it follows that the quasidihedral group QD_{2^n} is not a t-group.

Proposition 3.4. *Let G be the quasidihedral group QD_{2^n} and $d = 2^m$ be a divisor of 2^n . Then, the graph $\Gamma_d(G)$ is complete if and only if $d \neq 2$.*

Proof. First assume that $d \neq 2$. Then by Lemma 3.3, a subgroup of G of order $d = 2^m$ is either $H_1 = \langle a^{2^{n-m-1}} \rangle \cong C_{2^m}$ or conjugate to exactly one of $H_2 = \langle a^{2^{n-m}}, b \rangle$ or $H_3 = \langle a^{2^{n-m}}, ab \rangle$. Note that H_1 is a normal subgroup of QD_{2^n} and so its all NRTs are isomorphic to $QD_{2^n}/H_1 (\cong D_{2^{n-m}})$.

Now choose $S_2 = \{a^{2i+j}b^j \mid 0 \leq i < 2^{n-m-1}, j = 0, 1\}$ in $\mathcal{T}(QD_{2^n}, H_2)$ and $S_3 = \{a^{2i}b^j \mid 0 \leq i < 2^{n-m-1}, j = 0, 1\}$ in $\mathcal{T}(QD_{2^n}, H_3)$. Note that $\langle S_2 \rangle = \langle a^2, ab \rangle$ and $\langle S_3 \rangle = \langle a^2, b \rangle$. Then, $H_{S_2} = \langle S_2 \rangle \cap H_2 = \langle a^{2^{n-m}} \rangle \trianglelefteq \langle S_2 \rangle$ and $H_{S_3} = \langle S_3 \rangle \cap H_3 = \langle a^{2^{n-m}} \rangle \trianglelefteq \langle S_3 \rangle$. Therefore $G_{S_2} = G_{S_3} = \{1\}$ and hence S_2 and S_3 are groups.

Let \circ_2 denote the induced binary operation on S_2 as described in the Section 1. One can observe that, $(a^2)^{2^{n-m-1}} = (ab)^2 = (ab \circ_2 a^2)^2 = 1$. This implies that $S_2 \cong D_{2^{n-m}}$. One can similarly observe that $S_3 \cong D_{2^{n-m}}$. This shows that the graph $\Gamma_d(QD_{2^n})$ is complete.

Finally assume that $d = 2$. Then, a subgroup of G of order 2 is either $H_1 = \langle a^{2^{n-2}} \rangle$ or a conjugate to $H_2 = \langle b \rangle$. Since $H_1 \trianglelefteq G$, every NRT of H_1 in G is isomorphic to $G/H_1 \cong D_{2^{n-1}}$.

Let H be a non-normal subgroup of QD_{2^n} of order 2. Then, H is contained in $\langle a^2, b \rangle \cong D_{2^{n-1}}$ and H is a conjugate to the subgroup $\langle b \rangle$. Clearly the core $Core_G(H)$ of H in QD_{2^n} is trivial. Now let S be an NRT of H in QD_{2^n} . Then, the order of $H_S = H \cap \langle S \rangle$ is less than or equal to 2. If $|H_S| = 1$, then $S = \langle S \rangle$ is a subgroup of QD_{2^n} . Therefore S is equal to either $\langle a \rangle$ or $\langle a^2, ab \rangle \cong Q_{2^{n-1}}$. Finally if $|H_S| = 2$, then $H_S = H$ and $\langle S \rangle = G$. Therefore $G_S \cong H_S/Core_{H_S}(H_S) = H/Core_G(H) \cong H$. Hence S is not a group for G_S is nontrivial. Therefore $S \not\cong D_{2^{n-1}}$. \square

It can be trivially observed that the number of vertices in the graph $\Gamma_d(QD_{2^n})$ is equal to the number of subgroups of QD_{2^n} of order d and is given by

$$V_d(QD_{2^n}) = \begin{cases} 1 & \text{if } d = 1 \text{ or } d = 2^n. \\ 2^{n-2} + 1 & \text{if } d = 2. \\ 2^{n-m} + 1 & \text{if } d = 2^m \text{ with } 1 < m < n. \end{cases}$$

Proposition 3.5. *Let p and q be distinct odd primes. Then, a group of order either pq or $4p$ or $2pq$ is t -group.*

Proof. Observe that a nontrivial proper subgroup of a group of order pq is a Sylow subgroup. Hence any two subgroups of same order are adjacent in corresponding transiso graph.

By classification of groups of order $4p$ (see [1, p.132-137]), a non-abelian group of order $4p$ is isomorphic to exactly one of D_{4p}, Q_{4p} , the alternating group $Alt(4)$ (for $p = 3$), $C_p \rtimes C_4$ (for $p \equiv 1 \pmod{4}$). The groups D_{4p} and Q_{4p} are t -groups from the propositions 2.5 and 3.2. Since any two subgroups of the group $Alt(4)$ of equal order are conjugate therefore the group $Alt(4)$ is also a t -group.

Let H_1 and H_2 be two distinct subgroups of $C_p \rtimes C_4$ of order 2. Then, there exist unique Sylow 2-subgroup K_i of $C_p \rtimes C_4$ containing H_i where $i = 1, 2$. Since K_1 and K_2 are conjugate, the subgroups H_1 and H_2 are conjugate. So H_1 and H_2 are adjacent in $\Gamma_2(C_p \rtimes C_4)$.

A non-abelian group of order $2pq$ is isomorphic to exactly one of the groups $D_{2pq}, D_{2p} \times C_q, D_{2q} \times C_p$ and $C_2 \times (C_q \rtimes C_p)$, $(C_q \rtimes C_p) \rtimes C_2$ (when p divides $q - 1$) (see [4, p.50]). $D_{2p} \times C_q, D_{2q} \times C_p$ and $C_2 \times (C_q \rtimes C_p)$ are t -groups due to the Proposition 2.10. Order of the normalizer $N_G(H)$ of a Sylow p -subgroup H of $(C_q \rtimes C_p) \rtimes C_2$ is $2p$ and H is unique Sylow p -subgroup of $N_G(H)$. Since all Sylow p -subgroups are conjugate, therefore their normalizers are also conjugate. □

Proposition 3.6. *Let G be a non-abelian group of order $2p^2$ for some odd prime p . Then, the group G is t -group if and only if G is isomorphic to either the dihedral group D_{2p^2} or $(C_p \times C_p) \rtimes C_2$.*

Proof. It is well known that a non-abelian group of order $2p^2$ is isomorphic to exactly one of the groups $D_{2p^2}, (C_p \times C_p) \rtimes C_2$ and $C_p \times D_{2p}$ (see [1, p.132-137]). Let $G \cong (C_p \times C_p) \rtimes C_2 = \langle a, b, c \mid a^p, b^p, c^2, [a, b], (ac)^2, (bc)^2 \rangle$. Then, all subgroups of $\langle a, b \rangle \cong C_p \times C_p$ are normal in G and their quotients are dihedral groups D_{2p} . Hence $\Gamma_p(G)$ is a complete graph. Now $\Gamma_{2p}(G)$ is also complete there are several NRTs of a subgroup H of G order $2p$ which are isomorphic to the cyclic group of order p . So G is a t -group.

Now let $G \cong C_p \times D_{2p} = \langle a, b, c \mid a^p, b^p, c^2, [a, b], [a, c], (bc)^2 \rangle$. Then, it is obvious that $\langle a \rangle$ and $\langle b \rangle$ are normal subgroups of G of order p such that $G/\langle a \rangle \cong D_{2p}$ and $G/\langle b \rangle \cong C_{2p}$. Hence $\Gamma_p(G)$ is not a complete graph. □

4 Classification of t -groups of Order less than 32

Abelian t -groups are already determined by Proposition 2.3 which tells that a finite abelian group G is a t -group if and only if it is isomorphic to the direct sum of a cyclic group C and a direct sum A of some elementary abelian groups, where $|A|$ and $|C|$ are coprime.

We shall use the notation K_n for a complete graph on n vertices. By $K_m \sqcup K_n$ we mean a simple graph having two connected components K_m and K_n . By $(C_n)^r$ we mean $\underbrace{C_n \times C_n \times \cdots \times C_n}_{r \text{ times}}$. By

$[a, b]$ we mean the commutator $aba^{-1}b^{-1}$ of the elements a and b .

In the table 1, we have listed transiso graphs for all abelian groups of order less than 32. We have excluded the cases $d = 1$ and $|G|$, for $\Gamma_1(G) = \Gamma_{|G|}(G) = K_1$. We have used the software GAP [3] while determining some transiso graphs.

Non-abelian groups of the order 12, 20, 21, 28 and 30 are t-groups by Proposition 3.5 and a non-abelian t-group of the order 18 can be determined by Proposition 3.6. By Propositions 3.2 and 2.5, it is clear that the non-abelian groups of order 8 and $2p$ (for odd prime $p \leq 13$) are t-groups. In Propositions 4.1 and 4.3, we have determined non-abelian t-groups of the order 16 and 24 respectively. In the table 2, We have listed transiso graphs of the non-abelian groups of order less than 32.

We recall that a finite p -group P is p -central if each subgroup of P of order p is contained in the center $Z(P)$.

Proposition 4.1. *Let G be a non-abelian group of order 16. Then, the group G is a t-group if and only if G is isomorphic to either dihedral group D_{16} or dicyclic group Q_{16} .*

Proof. If G is a 2-central group, then it is isomorphic to one of the groups Q_{16} , $C_4 \rtimes C_4$ and $C_2 \times Q_8$ (see [11]). By Proposition 3.2, Q_{16} is a t-group. The group $C_4 \rtimes C_4 = \langle a, b \mid a^4, b^4, abab^{-1} \rangle$ has three normal subgroups $\langle a^2 \rangle$, $\langle b^2 \rangle$ and $\langle a^2b^2 \rangle$ of order 2 with quotient groups isomorphic to the groups $C_4 \times C_2$, D_8 and Q_8 respectively. Therefore the graph $\Gamma_2(C_4 \rtimes C_4)$ is not complete and hence $C_4 \rtimes C_4$ is not a t-group. The group $C_2 \times Q_8 = \langle a, b, c \mid a^2, b^4, b^2c^2, [a, b], [a, c], bcabc^{-1} \rangle$ is not a t-group, for it has three normal subgroups $\langle a \rangle$, $\langle b^2 \rangle$ and $\langle ab^2 \rangle$ of order 2 with quotient groups isomorphic to the groups Q_8 , $(C_2)^2 \times C_2$ and Q_8 respectively. Therefore $C_4 \rtimes C_4$ is not a t-group.

If G is a non 2-central group which is also a t-group, then $\Gamma_2(G)$ is a complete graph and hence by Proposition 2.6, G should be isomorphic to a nontrivial semidirect product $H \rtimes K$ of a non-normal subgroup H of G of order 2 and a normal subgroup K of G of order 8 such that for any normal subgroup L of G of order 2, K is isomorphic to G/L . By [11], we observe that there are five groups $(C_4 \times C_2) \rtimes_1 C_2$, $C_8 \rtimes C_2$, $QD_{16} = C_8 \rtimes_1 C_2$, $D_{16} = D_8 \rtimes C_2$ and $(C_4 \times C_2) \rtimes_2 C_2$ of required semidirect product type. Proposition 2.5 asserts that the group D_{16} is a t-group and the group QD_{16} is not a t-group by Proposition 3.4. The groups $(C_4 \times C_2) \rtimes_1 C_2$, $C_8 \rtimes C_2$ and $(C_4 \times C_2) \rtimes_2 C_2$ have normal subgroups of order 2 such that corresponding quotient groups are isomorphic to D_8 , $C_4 \times C_2$ and $(C_2)^3$ respectively (see [11]). Therefore these groups are not t-groups. \square

Lemma 4.2. *Let G be the group $C_2 \times Alt(4)$. Then, the graph $\Gamma_2(G)$ is not a complete graph.*

Proof. Let H be a non-normal subgroup of the group $G = C_2 \times Alt(4) = \langle a \rangle \times Alt(4)$ and $|H| = 2$. Then, the core $Core_G(H)$ of H in G is trivial. Let $S \in \mathcal{T}(G, H)$. Then, $H_S = \langle S \rangle \cap H$ and so $|H_S| \leq |H| = 2$. If $|H_S| = 1$, then S is a subgroup of G of order 12 i.e. $S = \{1\} \times Alt(4)$. If $|H_S| = 2$, then $H_S = H$ and so $\langle S \rangle = G$.

If H is a subgroup of G of order 2 contained in $\{1\} \times Alt(4)$, then there does not exist any NRT $S \in \mathcal{T}(G, H)$ such that $S \cong Alt(4)$, for otherwise $\langle S \rangle = S = \{1\} \times Alt(4)$ and $S \cap H = H$ which contradicts that S is an NRT.

It is easy to observe that an automorphism $f \in Aut(Alt(4))$ can be uniquely extended to an automorphism $\bar{f} \in Aut(G)$ such that $\bar{f}|_{Alt(4)} = f$ and $\bar{f}|_{\langle a \rangle} = \text{the identity map on } \langle a \rangle$.

Let us consider the subgroups $H = \langle (1, (1, 2)(3, 4)) \rangle$ and $K = \langle (a, (1, 2)(3, 4)) \rangle$ of the group G . Now if $H \sim_2 K$, then there exist $S \in \mathcal{T}(G, H)$ and $T \in \mathcal{T}(G, K)$ such that $S \cong K$. It is observed above that $S \not\cong Alt(4)$. Therefore $\langle S \rangle = \langle T \rangle = G$ and hence by Proposition 2.2, an

isomorphism $\phi : S \longrightarrow T$ can be extended to an automorphism $\bar{\phi}$ of G such that $\bar{\phi}(H) = K$ which is a contradiction, for there is no automorphism of G which maps H onto K . \square

Proposition 4.3. *Let G be a non-abelian group of order 24. Then, the group G is a t-group if and only if G is isomorphic to a semidirect product of two t-groups of coprime order except the groups $C_2 \times \text{Alt}(4)$ and $(C_2 \times C_6) \rtimes C_2$.*

Proof. We know that there are 12 non isomorphic non-abelian groups of order 24 (see [1, p.101-104]) and 8 of them are semidirect product of two t-groups of coprime order (see the table 2).

It is obvious that the groups $C_3 \rtimes C_8$ and $SL(2, 3)$ are t-groups, for any two subgroups of respective groups of equal order are conjugate. The groups Q_{24} and D_{24} are also t-groups by Propositions 3.2 and 2.5 respectively. By Proposition 2.10, we can see that the groups $C_3 \times D_8$, $C_3 \times Q_8$ and $C_2 \times D_{12} \cong (C_2)^2 \times D_6$ are t-groups. It is clear from [6, Example 2.2] that the symmetric group $Sym(4)$ is not a t-group. One can observe that $\langle a^2 \rangle \times \text{Alt}(3) \cong C_6$ and $\{1\} \times Sym(3)$ are normal subgroups of the group $\langle a \rangle \times Sym(3) \cong C_4 \times D_6$ such that their quotient groups are $(C_2)^2$ and C_4 respectively. So $\Gamma_6(C_4 \times D_6)$ is not a complete graph and hence the group $C_4 \times D_6$ is not a t-group. Similarly $C_2 \times Q_{12}$ is not a t-group since there are two normal subgroups of order 2 such that their quotient groups are D_{12} and Q_{12} . Now consider $G = (C_2 \times C_6) \rtimes C_2$. It has a normal subgroup H of order 2 such that $G/H \cong D_{12}$. Let K be a subgroup of G of order 2 contained in the subgroup isomorphic to D_{12} . Then, there is no NRT $S \in \mathcal{T}(G, H)$ such that $S \cong D_{12}$, for otherwise $S = \langle S \rangle$ and $S \cap H = H$ which contradicts the fact that S is an NRT. Therefore the group $(C_2 \times C_6) \rtimes C_2$ is not a t-group. Finally by Lemma 4.2, the group $C_2 \times \text{Alt}(4)$ is not a t-group. \square

5 Tables and Figures

Table 1: Transiso Graphs for Abelian Groups

| Order | Group G | Gap Id | Presentation | $\Gamma_d(G) = \Gamma_d$ | Is G a t-group? |
|-------|----------------------|--------|--|---|-----------------|
| 1 | C_1 | 1(1) | $\langle a \mid a^1 \rangle$ | | Yes |
| 2 | C_2 | 2(1) | $\langle a \mid a^2 \rangle$ | | Yes |
| 3 | C_3 | 3(1) | $\langle a \mid a^3 \rangle$ | | Yes |
| 4 | C_4 | 4(1) | $\langle a \mid a^4 \rangle$ | $\Gamma_2 = K_1$ | Yes |
| | $(C_2)^2$ | 4(2) | $\langle a, b \mid a^2, b^2, [a, b] \rangle$ | $\Gamma_2 = K_3$ | Yes |
| 5 | C_5 | 5(1) | $\langle a \mid a^5 \rangle$ | | Yes |
| 6 | C_6 | 6(2) | $\langle a \mid a^6 \rangle$ | $\Gamma_2 = \Gamma_3 = K_1$ | Yes |
| 7 | C_7 | 7(1) | $\langle a \mid a^7 \rangle$ | | Yes |
| 8 | C_8 | 8(1) | $\langle a \mid a^8 \rangle$ | $\Gamma_2 = \Gamma_4 = K_1$ | Yes |
| | $C_4 \times C_2$ | 8(2) | $\langle a, b \mid a^4, b^2, [a, b] \rangle$ | $\Gamma_2 = K_2 \sqcup K_1, \Gamma_4 = K_3$ | No |
| | $(C_2)^3$ | 8(5) | $\langle a, b, c \mid a^2, b^2, c^2, [a, b], [b, c], [a, c] \rangle$ | $\Gamma_2 = \Gamma_4 = K_7$ | Yes |
| 9 | C_9 | 9(1) | $\langle a \mid a^9 \rangle$ | $\Gamma_3 = K_1$ | Yes |
| | $(C_3)^2$ | 9(2) | $\langle a, b \mid a^3, b^3, [a, b] \rangle$ | $\Gamma_3 = K_4$ | Yes |
| 10 | C_{10} | 10(2) | $\langle a \mid a^{10} \rangle$ | $\Gamma_2 = \Gamma_5 = K_1$ | Yes |
| 11 | C_{11} | 11(1) | $\langle a \mid a^{11} \rangle$ | | Yes |
| 12 | C_{12} | 12(2) | $\langle a \mid a^{12} \rangle$ | $\Gamma_2 = \Gamma_3 = \Gamma_4 = \Gamma_6 = K_1$ | Yes |
| | $C_6 \times C_2$ | 12(5) | $\langle a, b \mid a^6, b^2, [a, b] \rangle$ | $\Gamma_2 = \Gamma_6 = K_3, \Gamma_3 = \Gamma_4 = K_1$ | Yes |
| 13 | C_{13} | 13(1) | $\langle a \mid a^{13} \rangle$ | | Yes |
| 14 | C_{14} | 14(2) | $\langle a \mid a^{14} \rangle$ | $\Gamma_2 = \Gamma_7 = K_1$ | Yes |
| 15 | C_{15} | 15(1) | $\langle a \mid a^{15} \rangle$ | $\Gamma_3 = \Gamma_5 = K_1$ | Yes |
| 16 | C_{16} | 16(1) | $\langle a \mid a^{16} \rangle$ | $\Gamma_2 = \Gamma_4 = \Gamma_8 = K_1$ | Yes |
| | $(C_4)^2$ | 16(2) | $\langle a, b \mid a^4, b^4, [a, b] \rangle$ | $\Gamma_2 = \Gamma_8 = K_3, \Gamma_4 = K_6 \sqcup K_1$ | No |
| | $C_8 \times C_2$ | 16(5) | $\langle a, b \mid a^8, b^2, [a, b] \rangle$ | $\Gamma_2 = \Gamma_4 = K_2 \sqcup K_1, \Gamma_8 = K_3$ | No |
| | $C_4 \times (C_2)^2$ | 16(10) | $\langle a, b, c \mid a^4, b^2, c^2, [a, b], [b, c], [a, c] \rangle$ | $\Gamma_2 = K_6 \sqcup K_1, \Gamma_4 = K_7 \sqcup K_4, \Gamma_8 = K_7$ | No |
| | $(C_2)^4$ | 16(14) | $\langle a_1, a_2, a_3, a_4 \mid a_i^2, [a_j, a_k] \ j \neq k \rangle$ | $\Gamma_2 = K_{15}, \Gamma_4 = K_{35}, \Gamma_8 = K_{15}$ | Yes |
| 17 | C_{17} | 17(1) | $\langle a \mid a^{17} \rangle$ | | Yes |
| 18 | C_{18} | 18(2) | $\langle a \mid a^{18} \rangle$ | $\Gamma_2 = \Gamma_3 = \Gamma_6 = \Gamma_9 = K_1$ | Yes |
| | $C_6 \times C_3$ | 18(5) | $\langle a, b \mid a^6, b^3, [a, b] \rangle$ | $\Gamma_2 = \Gamma_9 = K_1, \Gamma_3 = \Gamma_6 = K_4$ | Yes |
| 19 | C_{19} | 19(1) | $\langle a \mid a^{19} \rangle$ | | Yes |
| 20 | C_{20} | 20(2) | $\langle a \mid a^{20} \rangle$ | $\Gamma_2 = \Gamma_4 = \Gamma_5 = \Gamma_{10} = K_1$ | Yes |
| | $C_{10} \times C_2$ | 20(5) | $\langle a, b \mid a^{10}, b^2, [a, b] \rangle$ | $\Gamma_2 = \Gamma_{10} = K_3, \Gamma_4 = \Gamma_5 = K_1$ | Yes |
| 21 | C_{21} | 21(2) | $\langle a \mid a^{21} \rangle$ | $\Gamma_3 = \Gamma_7 = K_1$ | Yes |
| 22 | C_{22} | 22(2) | $\langle a \mid a^{22} \rangle$ | $\Gamma_2 = \Gamma_{11} = K_1$ | Yes |
| 23 | C_{23} | 23(1) | $\langle a \mid a^{23} \rangle$ | | Yes |
| 24 | C_{24} | 24(2) | $\langle a \mid a^{24} \rangle$ | $\Gamma_2 = \Gamma_3 = \Gamma_4 = \Gamma_6 = \Gamma_8 = \Gamma_{12} = K_1$ | Yes |
| | $C_{12} \times C_2$ | 24(9) | $\langle a, b \mid a^{12}, b^2, [a, b] \rangle$ | $\Gamma_2 = \Gamma_6 = K_2 \sqcup K_1, \Gamma_3 = \Gamma_8 = K_1, \Gamma_4 = \Gamma_{12} = K_3$ | No |
| | $C_6 \times (C_2)^2$ | 24(15) | $\langle a, b, c \mid a^6, b^2, c^2, [a, b], [b, c], [a, c] \rangle$ | $\Gamma_2 = \Gamma_4 = \Gamma_6 = \Gamma_{12} = K_7, \Gamma_3 = \Gamma_8 = K_1$ | Yes |
| 25 | C_{25} | 25(1) | $\langle a \mid a^{25} \rangle$ | $\Gamma_5 = K_1$ | Yes |
| | $(C_5)^2$ | 25(2) | $\langle a, b \mid a^5, b^5, [a, b] \rangle$ | $\Gamma_5 = K_6$ | Yes |
| 26 | C_{26} | 26(2) | $\langle a \mid a^{26} \rangle$ | $\Gamma_2 = \Gamma_{13} = K_1$ | Yes |
| 27 | C_{27} | 27(1) | $\langle a \mid a^{27} \rangle$ | $\Gamma_3 = \Gamma_9 = K_1$ | Yes |
| | $C_9 \times C_3$ | 27(2) | $\langle a, b \mid a^9, b^3, [a, b] \rangle$ | $\Gamma_3 = K_3 \sqcup K_1, \Gamma_9 = K_4$ | No |
| | $(C_3)^3$ | 27(5) | $\langle a, b, c \mid a^3, b^3, c^3, [a, b], [b, c], [a, c] \rangle$ | $\Gamma_3 = \Gamma_9 = K_{13}$ | Yes |
| 28 | C_{28} | 28(2) | $\langle a \mid a^{28} \rangle$ | $\Gamma_2 = \Gamma_4 = \Gamma_7 = \Gamma_{14} = K_1$ | Yes |
| | $C_{14} \times C_2$ | 28(4) | $\langle a, b \mid a^{14}, b^2, [a, b] \rangle$ | $\Gamma_2 = \Gamma_{14} = K_3, \Gamma_4 = \Gamma_7 = K_1$ | Yes |
| 29 | C_{29} | 29(1) | $\langle a \mid a^{29} \rangle$ | | Yes |
| 30 | C_{30} | 30(4) | $\langle a \mid a^{30} \rangle$ | $\Gamma_2 = \Gamma_3 = \Gamma_5 = \Gamma_6 = \Gamma_{10} = \Gamma_{15} = K_1$ | Yes |
| 31 | C_{31} | 31(1) | $\langle a \mid a^{31} \rangle$ | | Yes |

Table 2: Transiso graphs for Non-abelian groups

| Order | Group G | Gap Id | Presentation | $\Gamma_d(G) = \Gamma_d$ | Is G a t-group? |
|-------|--|--------|---|---|-----------------|
| 6 | D_6 | 6(1) | $\langle a, b \mid a^3, b^2, (ab)^2 \rangle$ | $\Gamma_2 = K_3, \Gamma_3 = K_1$ | Yes |
| 8 | D_8 | 8(3) | $\langle a, b \mid a^4, b^2, (ab)^2 \rangle$ | $\Gamma_2 = K_5, \Gamma_4 = K_3$ | Yes |
| | Q_8 | 8(4) | $\langle a, b \mid a^4, a^2b^2, abab^{-1} \rangle$ | $\Gamma_2 = K_1, \Gamma_4 = K_3$ | Yes |
| 10 | D_{10} | 10(1) | $\langle a, b \mid a^5, b^2, (ab)^2 \rangle$ | $\Gamma_2 = K_5, \Gamma_5 = K_1$ | Yes |
| 12 | Q_{12} | 12(1) | $\langle a, b \mid a^6, a^3b^{-2}, abab^{-1} \rangle$ | $\Gamma_2 = \Gamma_3 = \Gamma_6 = K_1, \Gamma_4 = K_3$ | Yes |
| | $Alt(4)$ | 12(3) | $\langle a, b \mid a^3, b^2, (ab)^3 \rangle$ | $\Gamma_2 = K_3, \Gamma_3 = K_4, \Gamma_4 = K_1$ | Yes |
| | D_{12} | 12(4) | $\langle a, b \mid a^6, b^2, (ab)^2 \rangle$ | $\Gamma_2 = K_7, \Gamma_3 = K_1, \Gamma_4 = \Gamma_6 = K_3$ | Yes |
| 14 | D_{14} | 14(1) | $\langle a, b \mid a^7, b^2, (ab)^2 \rangle$ | $\Gamma_2 = K_7, \Gamma_7 = K_1$ | Yes |
| 16 | $(C_4 \times C_2) \rtimes_1 C_2$ | 16(3) | $\langle a, b \mid a^2, b^4, (ab)^4, [a, b]^2, [a, b^2] \rangle$ | $\Gamma_2 = K_2 \sqcup K_5$ (Figure 1), $\Gamma_4 = \text{Figure 2}, \Gamma_8 = K_3$ | No |
| | $C_4 \rtimes C_4$ | 16(4) | $\langle a, b \mid a^4, b^4, a^2(ab)^2 \rangle$ | $\Gamma_2 = K_1 \sqcup K_1 \sqcup K_1, \Gamma_4 = \text{Figure 3}, \Gamma_8 = K_3$ | No |
| | $C_8 \rtimes C_2$ | 16(6) | $\langle a, b \mid a^8, b^2, (ba)^2a^2 \rangle$ | $\Gamma_2 = K_1 \sqcup K_2, \Gamma_4 = K_1 \sqcup K_2, \Gamma_8 = K_3$ | No |
| | D_{16} | 16(7) | $\langle a, b \mid a^8, b^2, (ab)^2 \rangle$ | $\Gamma_2 = K_9, \Gamma_4 = K_5, \Gamma_8 = K_3$ | Yes |
| | QD_{16} | 16(8) | $\langle a, b \mid a^8, b^2, baba^{-3} \rangle$ | $\Gamma_2 = K_1 \sqcup K_4, \Gamma_4 = K_5, \Gamma_8 = K_3$ | No |
| | Q_{16} | 16(9) | $\langle a, b \mid a^8, a^4b^{-2}, abab^{-1} \rangle$ | $\Gamma_2 = K_1, \Gamma_4 = K_5, \Gamma_8 = K_3$ | Yes |
| | $C_2 \times D_8$ | 16(11) | $\langle a, b, c \mid a^2, b^4, c^2, (bc)^2, [a, b], [a, c] \rangle$ | $\Gamma_2 = \text{Figure 2}, \Gamma_4 = K_{15}, \Gamma_8 = K_7$ | No |
| | $C_2 \times Q_8$ | 16(12) | $\langle a, b, c \mid a^2, b^4, b^2c^2, [a, b], [a, c], bc bc^{-1} \rangle$ | $\Gamma_2 = K_1 \sqcup K_2, \Gamma_4 = \Gamma_8 = K_7$ | No |
| | $(C_4 \times C_2) \rtimes_2 C_2$ | 16(13) | $\langle a, b, c \mid a^2, b^2, c^4, [a, b]c^2, [a, c], [b, c] \rangle$ | $\Gamma_2 = K_1 \sqcup K_6, \Gamma_4 = \Gamma_8 = K_7$ | No |
| 18 | D_{18} | 18(1) | $\langle a, b \mid a^9, b^2, (ab)^2 \rangle$ | $\Gamma_2 = K_9, \Gamma_6 = K_3, \Gamma_3 = \Gamma_9 = K_1$ | Yes |
| | $C_3 \times D_6$ | 18(3) | $\langle a, b, c \mid a^3, b^3, c^2, (bc)^2, [a, b], [a, c] \rangle$ | $\Gamma_2 = K_3, \Gamma_3 = \text{Figure 4}, \Gamma_6 = K_4, \Gamma_9 = K_1$ | No |
| | $(C_3 \times C_3) \rtimes C_2$ | 18(4) | $\langle a, b, c \mid a^3, b^3, c^2, [a, b], (ac)^2, (bc)^2 \rangle$ | $\Gamma_2 = K_9, \Gamma_3 = K_4, \Gamma_6 = K_{12}, \Gamma_9 = K_1$ | Yes |
| 20 | Q_{20} | 20(1) | $\langle a, b \mid a^{10}, a^5b^{-2}, abab^{-1} \rangle$ | $\Gamma_2 = \Gamma_5 = \Gamma_{10} = K_1, \Gamma_4 = K_5$ | Yes |
| | $C_5 \times C_4$ | 20(3) | $\langle a, b \mid a^5, b^4, [a, b]a^{-1} \rangle$ | $\Gamma_2 = \Gamma_4 = K_5, \Gamma_5 = \Gamma_{10} = K_1$ | Yes |
| | D_{20} | 20(4) | $\langle a, b \mid a^{10}, b^2, (ab)^2 \rangle$ | $\Gamma_2 = K_{11}, \Gamma_4 = K_5, \Gamma_5 = K_1, \Gamma_{10} = K_3$ | Yes |
| 21 | $C_7 \times C_3$ | 21(1) | $\langle a, b \mid a^3, b^7, [b, a]b^{-1} \rangle$ | $\Gamma_3 = K_7, \Gamma_7 = K_1$ | Yes |
| 22 | D_{22} | 22(1) | $\langle a, b \mid a^{11}, b^2, (ab)^2 \rangle$ | $\Gamma_2 = K_{11}, \Gamma_{11} = K_1$ | Yes |
| 24 | $C_3 \times C_8$ | 24(1) | $\langle a, b \mid a^3, b^8, abab^{-1} \rangle$ | $\Gamma_2 = \Gamma_3 = \Gamma_4 = K_1, \Gamma_8 = K_3, \Gamma_6 = \Gamma_{12} = K_1$ | Yes |
| | $SL(2, 3) = Q_8 \times C_3$ | 24(3) | $\langle a, b \mid a^3, b^4, (ab)^4, ab(ab^{-1})^2, [b^2, a] \rangle$ | $\Gamma_2 = \Gamma_8 = K_1, \Gamma_4 = K_3, \Gamma_3 = \Gamma_6 = K_4$ | Yes |
| | $Q_{24} = C_3 \times Q_8$ | 24(4) | $\langle a, b \mid a^{12}, a^6b^{-2}, abab^{-1} \rangle$ | $\Gamma_2 = \Gamma_3 = \Gamma_6 = K_1, \Gamma_4 = K_7, \Gamma_8 = \Gamma_{12} = K_3$ | Yes |
| | $C_4 \times D_6 = C_3 \rtimes_1 (C_4 \times C_2)$ | 24(5) | $\langle a, b, c \mid a^4, b^3, c^2, (bc)^2, [a, b], [a, c] \rangle$ | $\Gamma_2 = K_1 \sqcup K_6, \Gamma_3 = K_1, \Gamma_4 = K_7, \Gamma_6 = K_2 \sqcup K_1, \Gamma_8 = \Gamma_{12} = K_3$ | No |
| | $D_{24} = C_3 \rtimes_1 D_8$ | 24(6) | $\langle a, b \mid a^{12}, b^2, (ab)^2 \rangle$ | $\Gamma_2 = K_{13}, \Gamma_4 = K_7, \Gamma_3 = K_1, \Gamma_6 = K_5, \Gamma_8 = \Gamma_{12} = K_3$ | Yes |
| | $C_2 \times Q_{12} = C_3 \rtimes_2 (C_4 \times C_2)$ | 24(7) | $\langle a, b, c \mid a^2, b^6, b^3c^2, bc bc^{-1}, [a, b], [a, c] \rangle$ | $\Gamma_2 = K_1 \sqcup K_2 = \Gamma_6, \Gamma_3 = K_1, \Gamma_4 = K_7, \Gamma_8 = \Gamma_{12} = K_3$ | No |
| | $(C_2 \times C_6) \rtimes C_2 = C_3 \rtimes_2 D_8$ | 24(8) | $\langle a, b, c \mid a^2, b^2, c^3, [b, c], (ab)^4, (ac)^2 \rangle$ | $\Gamma_2 = \text{Figure 5}, \Gamma_6 = K_5, \Gamma_4 = K_7, \Gamma_3 = K_1, \Gamma_8 = \Gamma_{12} = K_3$ | No |
| | $C_3 \times D_8$ | 24(10) | $\langle a, b, c \mid a^3, b^4, c^2, (bc)^2, [a, b], [a, c] \rangle$ | $\Gamma_2 = K_5 = \Gamma_6, \Gamma_4 = K_3, \Gamma_3 = K_1 = \Gamma_8, \Gamma_{12} = K_3$ | Yes |
| | $C_3 \times Q_8$ | 24(11) | $\langle a, b, c \mid a^3, b^4, b^2c^2, bc bc^{-1}, [a, b], [a, c] \rangle$ | $\Gamma_2 = \Gamma_3 = \Gamma_6 = \Gamma_8 = K_1, \Gamma_4 = K_3 = \Gamma_{12}$ | Yes |
| | $Sym(4)$ | 24(12) | $\langle a, b \mid a^2, b^3, (ab)^4 \rangle$ | $\Gamma_2 = K_3 \sqcup K_6, \Gamma_3 = \Gamma_6 = K_4, \Gamma_4 = K_1 \sqcup K_3 \sqcup K_3, \Gamma_8 = K_3, \Gamma_{12} = K_1$ | No |
| | $C_2 \times Alt(4) = (C_2)^3 \rtimes C_3$ | 24(13) | $\langle a, b, c \mid a^2, b^3, c^2, (bc)^3, [a, b], [a, c] \rangle$ | $\Gamma_2 = K_3 \sqcup K_4, \Gamma_3 = \Gamma_6 = K_4, \Gamma_4 = \text{Figure 6}, \Gamma_8 = \Gamma_{12} = K_1$ | No |
| | $C_2 \times D_{12} = C_3 \rtimes (C_2)^3$ | 24(14) | $\langle a, b, c \mid a^2, b^6, c^2, (bc)^2, [a, b], [a, c] \rangle$ | $\Gamma_2 = K_{15}, \Gamma_3 = K_1, \Gamma_4 = K_{19}, \Gamma_8 = K_3, \Gamma_6 = \Gamma_{12} = K_7$ | Yes |
| 26 | D_{26} | 26(1) | $\langle a, b \mid a^{13}, b^2, (ab)^2 \rangle$ | $\Gamma_2 = K_{13}, \Gamma_7 = K_1$ | Yes |
| 27 | $(C_3 \times C_3) \rtimes C_3$ | 27(3) | $\langle a, b \mid a^3, b^3, (ab)^3, (a^2b)^3 \rangle$ | $\Gamma_3 = K_{13}, \Gamma_9 = K_4$ | Yes |
| | $C_9 \times C_3$ | 27(4) | $\langle a, b \mid a^9, b^3, [a, b]a^{-3} \rangle$ | $\Gamma_3 = K_1 \sqcup K_3, \Gamma_9 = K_4$ | No |
| 28 | Q_{28} | 28(1) | $\langle a, b \mid a^{14}, a^7b^{-2}, abab^{-1} \rangle$ | $\Gamma_2 = \Gamma_7 = \Gamma_{14} = K_1, \Gamma_4 = K_7$ | Yes |
| | D_{28} | 28(3) | $\langle a, b \mid a^{14}, b^2, (ab)^2 \rangle$ | $\Gamma_2 = K_{15}, \Gamma_4 = K_7, \Gamma_7 = K_1, \Gamma_{14} = K_3$ | Yes |
| 30 | $C_5 \times D_6$ | 30(1) | $\langle a, b, c \mid a^5, b^3, c^2, (bc)^2, [a, b], [a, c] \rangle$ | $\Gamma_3 = \Gamma_5 = \Gamma_6 = \Gamma_{15} = K_1, \Gamma_2 = \Gamma_{10} = K_3$ | Yes |
| | $C_3 \times D_{10}$ | 30(2) | $\langle a, b, c \mid a^3, b^5, c^2, (bc)^2, [a, b], [a, c] \rangle$ | $\Gamma_3 = \Gamma_5 = \Gamma_{10} = \Gamma_{15} = K_1, \Gamma_2 = \Gamma_6 = K_5$ | Yes |
| | D_{30} | 30(3) | $\langle a, b \mid a^{15}, b^2, (ab)^2 \rangle$ | $\Gamma_2 = K_{15}, \Gamma_6 = K_5, \Gamma_{10} = K_3, \Gamma_3 = \Gamma_5 = \Gamma_{15} = K_1$ | Yes |

Figure 1: Transiso graph $\Gamma_2((C_4 \times C_2) \rtimes C_2) = K_2 \sqcup K_5$

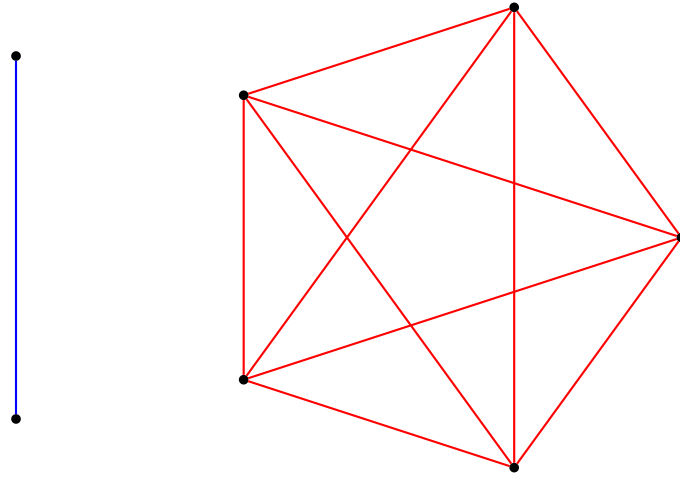


Figure 2: Transiso graph $\Gamma_4((C_4 \times C_2) \rtimes C_2)$ or Transiso graph $\Gamma_2(C_2 \times D_8)$

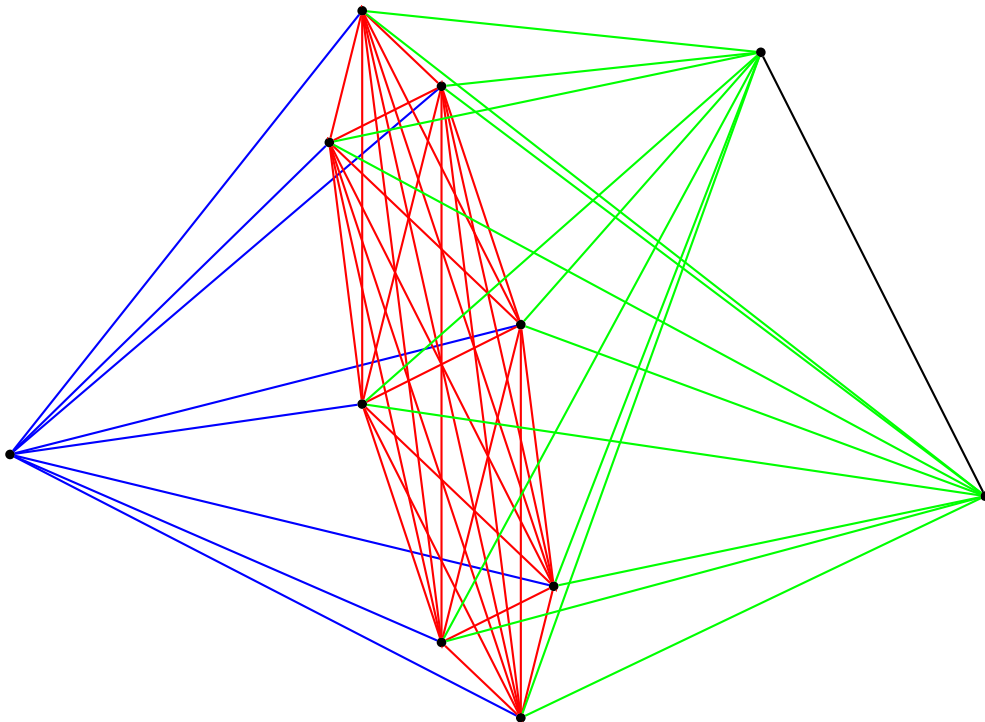


Figure 3: Transiso graph $\Gamma_4(C_4 \rtimes C_4)$

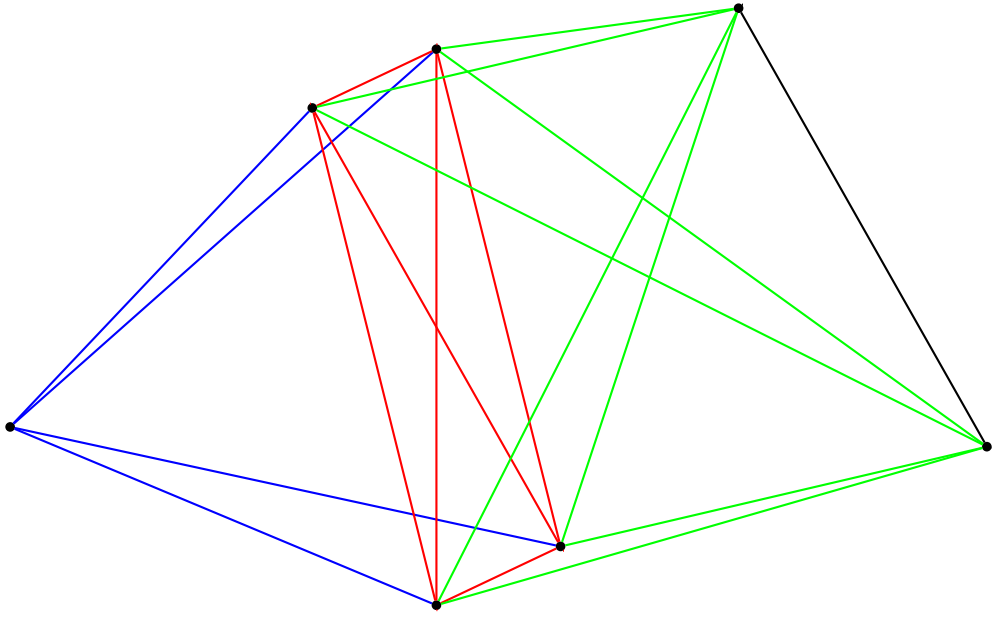


Figure 4: Transiso graph $\Gamma_3(C_3 \times D_6)$

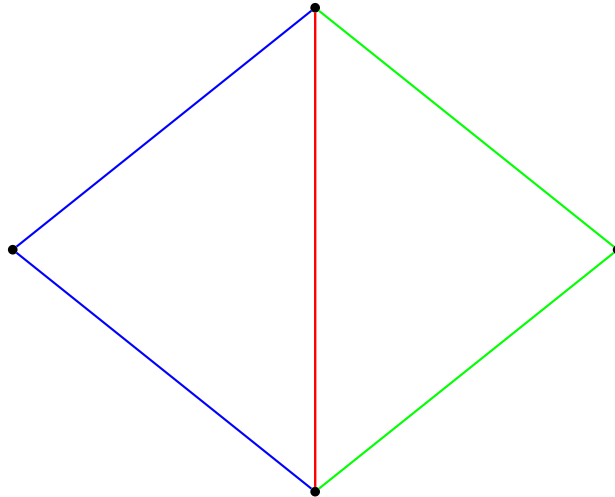


Figure 5: Transiso graph $\Gamma_2((C_2 \times C_6) \rtimes C_2)$

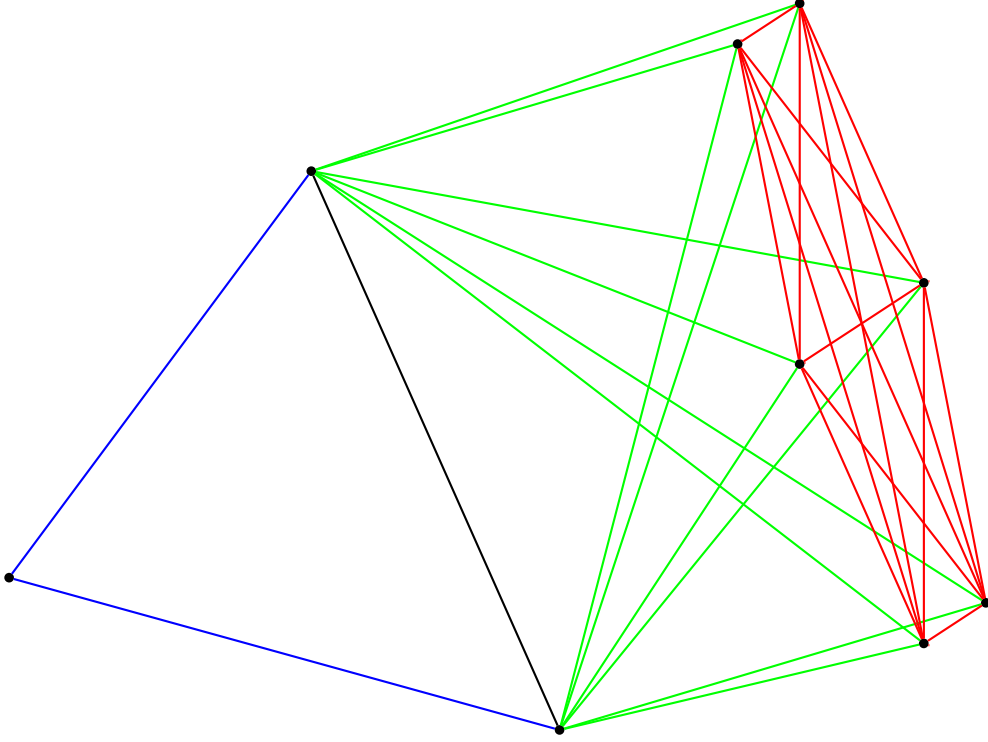
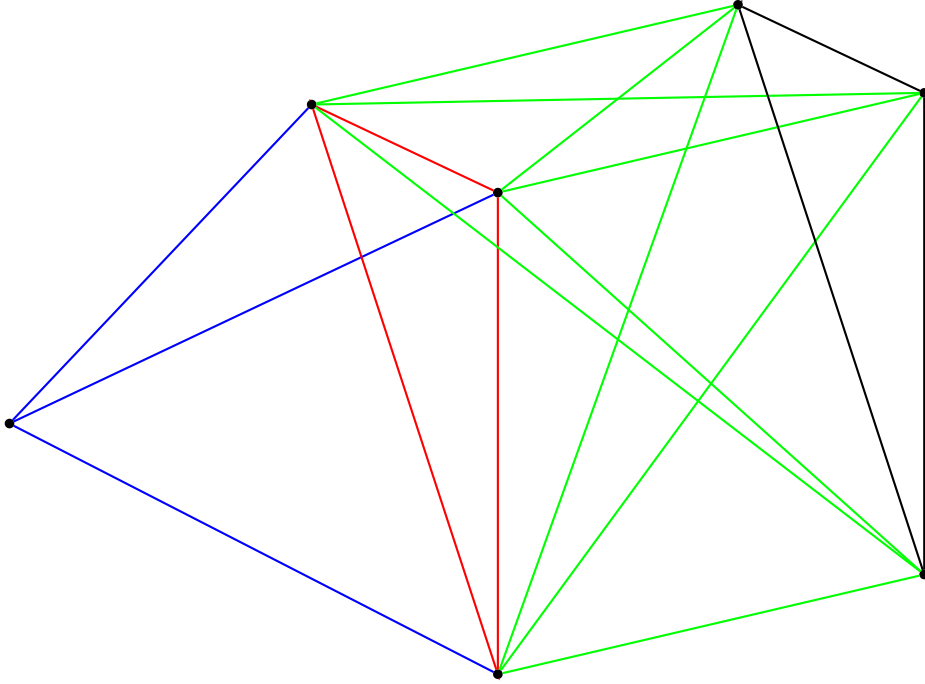


Figure 6: Transiso graph $\Gamma_4(C_2 \times Alt(4))$



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References

- [1] W. Burnside, *Theory of groups of finite order*, (1897), Cambridge University Press.
- [2] P. J. Cameron, preprint available at
<http://www.maths.qmul.ac.uk/~pjc/preprints/transgenic.pdf>.
- [3] The GAP Group, *GAP-Groups, Algorithms, and Programming, Version 4.7.4, 2014*,
<http://www.gap-system.org>.
- [4] M. Ghorbani and F. N. Larki, *Automorphism Group of Groups of Order pqr* , Algebraic Structures and Their Applications (2014), 49-56.
- [5] D. Gorenstein, *Finite Groups*, AMS Chelsea Publishing (2007).
- [6] V. Kakkar and L. K. Mishra, *On Transiso Graph*, Asian European Journal of Mathematics, (2015). <http://dx.doi.org/10.1142/S1793557115500709>.
- [7] R. Lal, Transversals in Groups, *J. Algebra* **181** (1996) 70-81.
- [8] S. Roman, *Fundamentals of Group Theory: An Advanced Approach* (2012) (New York: Birkhauser).
- [9] J. D. H. Smith, *An Introduction to Quasigroups and Their Representations* (2007) (Boca Raton, FL: Chapman and Hall/CRC).
- [10] M. Suzuki, *Group Theory I* (1982) (New York: Springer-Verlag).
- [11] M. Wild, *The groups of order sixteen made easy*, The American Mathematical Monthly, 112, 20-31, 2005.